Scholarly Research Journal for Interdisciplinary Studies,
Online ISSN 2278-8808, SJIF 2019 = 6.380, www.srjis.com
PEER REVIEWED & REFEREED JOURNAL, NOV-DEC, 2020, VOL- 8/62

STUDY OF BRINKMAN-BÉNARD CONVECTION IN THE CHAOTIC REGION USING LYAPUNOV-EXPONENT PLOTS AND BIFURCATION DIAGRAMS

B. Manjunath[1] & Roopa. D. S.[2]

[1]*M.Sc.,M.Phil., Assistant Proffesor, Mathematics, GFGC, MBL.*

Email: manjumaths6004@gmail.com

[2]*M.Sc., Guest Faculty Department of Department of Mathematics, GFGC, MBL*

Introduction

1 Mathematical Formulation

An infinite extent horizontal liquid saturated porous layer of thickness, d, whose lower and upper bounding planes are at z=0 and z=d respectively is considered(see Fig.1). The liquid is assumed to be a viscous, Newtonian liquid. The upper and lower boundaries are maintained at constant temper- atures T_0 and $T_0 + \Delta T$ ($\Delta T > 0$) respectively. For mathematical tractability we confine ourselves to two-dimensional longitudinal rolls so that all physical quantities are independent of x, a horizontal co-ordinate. The region of inter- est is $R = \{(y, z)/ -\infty < y < \infty, 0 \le z \le d\}$. The boundaries are assumedto be stress-free and isothermal. In this project we assume the dynamic co- efficient of viscosity of the liquid, μ_l, and effective thermal diffusivity of the liquid, α_l, to be constants. However, the density of the carrier liquid, ρ_l, is temperature-dependent.

We assume that the Boussinesq approximation is valid. The governing equa- tions describing the Rayleigh-Bénard-Brinkman instability situation in a Newtonian liquid saturated porous medium are:

Conservation of Mass

$$\nabla . \mathbf{q} = \mathbf{0}, \qquad (1)$$

Conservation of momentum

$$\rho_0 \left[\frac{\partial \mathbf{q}}{\partial t} + (\mathbf{q}.\nabla)\mathbf{q} \right] = -\nabla p + \mu' \nabla^2 \mathbf{q} + \rho \, \mathbf{g} - \frac{\mu_l}{k} \mathbf{q}, \qquad (2)$$

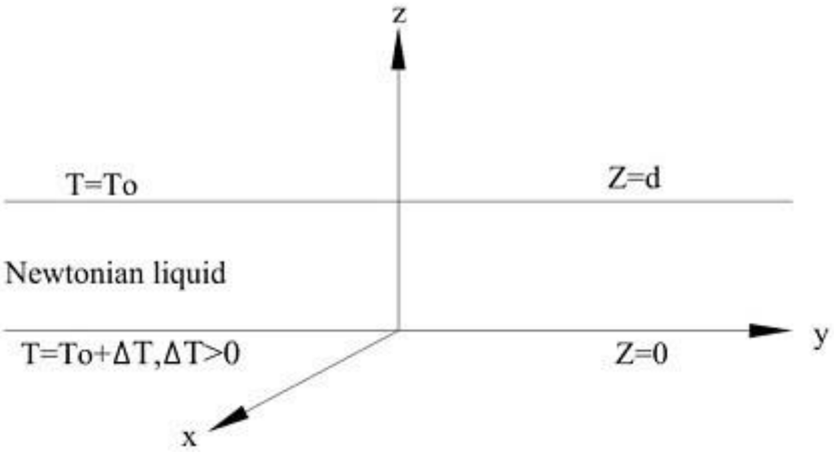

Figure 1: Physical configuration

Conservation of energy

$$\frac{\partial T}{\partial t} + (\mathbf{q}.\nabla)T = \chi\nabla^2 T, \qquad (3)$$

Equation of state

$$\rho(T) = \rho_0[1 - \alpha(T - T_0)], \qquad (4)$$

where $\mathbf{q} = (0, v, w)$ is the velocity vector, v is the horizontal component of velocity, w is the vertical component of velocity, y is the horizontal coordi- nate, z is the vertical coordinate, T_0 is the reference temperature, ρ_l is the density of the liquid at $T = T_0$, t is the time, p is the pressure, μ is the dynamic coefficient of viscosity of the liquid, μ' is the effective viscosity, α_l is the coefficient of thermal expansion of the liquid, T is the dimensional temperature, $\mathbf{g} = g\hat{k}$ is the acceleration due to gravity, χ is the effective thermal diffusivity of the liquid and k_l is the thermal conductivity of the liquid.

Since we are considering two-dimensional convective motion, we have

$$\mathbf{q} = v(y,z,t)\hat{\mathbf{j}} + w(y,z,t)\hat{\mathbf{k}},\ \ T = T(y,z,t),\ \ \rho = \rho(y,z,t),\ \ p = p(y,z,t). \quad (5)$$

Sing Eq. (5) in Eqs. (1)-(4), we get

$$\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0, \tag{6}$$

$$\rho_0\left(\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=-\frac{\partial p}{\partial y}+\mu'\left(\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)-\frac{\mu}{k}v, \tag{7}$$

$$\rho_0\left(\frac{\partial w}{\partial t}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=-\frac{\partial p}{\partial z}+\mu'\left(\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)-\frac{\mu}{k}w-\rho g, \tag{8}$$

$$\frac{\partial T}{\partial t}+v\frac{\partial T}{\partial y}+w\frac{\partial T}{\partial z}=\chi\left(\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right). \tag{9}$$

The expression for the effective viscosity and effective thermal conductivity are appropriate for spherical-particles suspended in a carrier liquid. Taking the velocity, temperature and density fields in the quisecent basic state to be $q_b(z)=(0,0)$, $T_b(z)$ and $\rho_b(z)$, we obtain the quiescent state solution in the form:

$$\mathbf{q}_b=(0,0),\ T_b=T_0+\Delta T\left(1-\frac{z}{d}\right),\ p_b=-g\int\rho_b(T_b)\,dz+c, \tag{10}$$

where c is the constant of integration. The quiescent basic state is motionlessand, in fact, the initial state of the system. On the quiescent basic state we superimpose perturbation in the form:

$$v=v_b+v',\ w=w_b+w',\ T=T_b+T',\ \rho=\rho_b+\rho',\ p=p_b+p', \tag{11}$$

where the prime indicates a perturbed quantity.

Now, the governing equations (6)-(9) become

$$\frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0, \qquad (12)$$

$$\rho_0\left(\frac{\partial v'}{\partial t} + v'\frac{\partial v'}{\partial y} + w'\frac{\partial v'}{\partial z}\right) = -\frac{\partial p'}{\partial y} + \mu'\left(\frac{\partial^2 v'}{\partial y^2} + \frac{\partial^2 v'}{\partial z^2}\right) - \frac{\mu}{k}v', \qquad (13)$$

$$\rho_0\left(\frac{\partial w'}{\partial t} + v'\frac{\partial w'}{\partial y} + w'\frac{\partial w'}{\partial z}\right) = -\frac{\partial p'}{\partial z} + \mu'\left(\frac{\partial^2 w'}{\partial y^2} + \frac{\partial^2 w'}{\partial z^2}\right) - \frac{\mu}{k}w' - \rho'(T')g, \qquad (14)$$

$$\frac{\partial T'}{\partial t} + v'\frac{\partial T'}{\partial y} + w'\frac{\partial T'}{\partial z} - w'\frac{\Delta T}{d} = \chi\left(\frac{\partial^2 T'}{\partial y^2} + \frac{\partial^2 T'}{\partial z^2}\right), \qquad (15)$$

$$\rho_b(T_b) + \rho'(T') = \rho_0[1 - \alpha(T_b + T' - T_0)]. \qquad (16)$$

But

$$\rho_b(T_b) = \rho_0[1 - \alpha(T_b - T_0)]. \qquad (17)$$

Using Eq. (17) in Eq. (16), we get

$$\rho'(T') = -\alpha\rho_0 T'. \qquad (18)$$

For simplicity we neglect the primes in Eqs. (12)-(18) to get

$$+\frac{\partial v}{} \quad \frac{\partial w}{} \qquad \frac{\partial y}{\partial y^2} + \frac{\partial z}{\partial z^2} - \frac{}{k}v, \qquad (20)$$

$$= 0, \tag{19}$$

$$\rho_0\left(\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = -\frac{\partial p}{\partial y} + \mu'\left(\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) - \frac{\mu}{k}v,$$

$$\rho_0\left(\frac{\partial w}{\partial t} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu'\left(\frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) - \frac{\mu}{k}w + \alpha\rho_0 T g, \tag{21}$$

$$\frac{\partial T}{\partial t} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z} + w\left(-\frac{\Delta T}{d}\right) = \chi\left(\frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right). \tag{22}$$

Equations (19)-(22) are four equations in the four unknowns v, w, p and T.

Differentiating Eq. (20) with respect to ‘ z ’, we get

$$\rho_0\left(\frac{\partial}{\partial t}\frac{\partial v}{\partial z} + \frac{\partial}{\partial z}\left(v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right)\right) = -\frac{\partial^2 p}{\partial y\partial z} + \mu'\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\frac{\partial v}{\partial z} - \frac{\mu}{k}\frac{\partial v}{\partial z}. \tag{23}$$

Differentiating Eq. (21) with respect to ‘y’, we get

$$\rho_0\left(\frac{\partial}{\partial t}\frac{\partial w}{\partial y} + \frac{\partial}{\partial y}\left(v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right)\right) = -\frac{\partial^2 p}{\partial z\partial y} + \mu'\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\frac{\partial w}{\partial y}$$

$\partial y \quad \partial y \quad = - \partial y \partial z + \mu \quad \partial y^2 \quad + \quad \partial z^2$

$\partial z \quad \mu\, \partial w \quad \partial$

y

∂T

$— k\, \partial y + \alpha \rho_0 g\, \partial y \, .$

(24)

Substracting Eq. (23) from Eq. (24), we get

$$\rho_0 \frac{\partial}{\partial t}\left(\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\right) + \rho_0 \frac{\partial}{\partial z}\left(v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) - \rho_0 \frac{\partial}{\partial y}\left(v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) \tag{25}$$

$$= \mu \left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\left(\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\right) - \frac{\mu}{k}\left(\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\right) - \alpha \rho_0 g \frac{\partial T}{\partial y} \, .$$

We now reduce the number of dependent variables by making use of the stream function.

Let us introduce the stream function $\psi(y, z, t)$ in the form

$$v = \frac{\partial \psi}{\partial z}, \quad w = -\frac{\partial \psi}{\partial y} . \tag{26}$$

Noting that

$$\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y} = \nabla^2 \psi \tag{27}$$

and substituting Eqs. (26)-(27) in Eq. (25), we get

$\partial \quad 2 \quad \partial T \quad J \quad 4 \quad \mu \quad 2 \quad \partial(\psi, \nabla^2 \psi)$

$$\rho_0 \frac{\partial}{\partial t}(\nabla \psi) = -\alpha\rho_0 g \frac{\partial}{\partial y} + \mu \nabla \psi - \frac{}{k}\nabla \psi + \rho_0 \frac{}{\partial(y,z)}. \quad (28)$$

Using the stream function from Eq. (26) in Eq. (22), we get

$$\frac{\partial T}{\partial t} = -\frac{\Delta T}{d}\frac{\partial \psi}{\partial y} + \chi Q^2 T + \frac{\partial(\psi, T)}{\partial(y, z)}. \quad (29)$$

Equations (28) and (29) are the governing stability equations for Rayleigh-Bénard-Brinkman convection. There are two equations in the two unknows ψ and T.

1.1 Non-Dimensionalization

We non-dimensionalize Eqs. (28) and (29) using the following definition:

$$(Y, Z) = \frac{y}{d}, \frac{z}{d}, \quad \tau = \frac{t\chi}{d^2}, \quad \Psi = \frac{\psi}{\chi}, \quad \Theta = \frac{T}{\Delta T}. \quad (30)$$

Using the Eq. (30) in Eqs. (28) and (29), we obtain the dimensionless form of the vorticity and heat transport equations in the form:

$$\frac{1}{\Pr}\frac{\partial}{\partial \tau}(\nabla^2 \Psi) = -\mathrm{Ra}\frac{\partial \Theta}{\partial Y} + \Lambda Q^4 \Psi - \sigma^2 \nabla^2 \Psi + \frac{1}{\Pr}\frac{\partial(\Psi, \nabla^2\Psi)}{\partial(Y, Z)}, \quad (31)$$

$$\frac{\partial \Theta}{\partial \tau} = -\frac{\partial \Psi}{\partial Y} + \nabla^2\Theta + \frac{\partial(\Psi, \Theta)}{\partial(Y, Z)}, \quad (32)$$

where $\Pr = \frac{\mu}{\rho\chi}$ is the Prandtl number,

$\mathrm{Ra} = \frac{\alpha\rho_0 g d^3 \Delta T}{\mu\chi}$ is the Rayleigh number,

$\frac{\mu'}{\mu}$ is the ratios of viscosity ,

$\frac{d^2}{K}$ is the porous parameter.

Equations (31) and (32) are the non-dimensional versions of the govern- ing stability equations for Rayleigh-Bénard convection. These equations are solved using the boundary / periodicity conditions:

$$\Psi = \frac{\partial^2 \Psi}{\partial Z^2} = \frac{\partial \Psi}{\partial Y} = \Theta = 0 \text{ at } Z = 0, 1, \qquad (33)$$

$$\Psi\left(Y \pm \frac{2\pi}{\pi k_c}, Z\right) = \Psi(Y, Z), \quad \Theta\left(Y \pm \frac{2\pi}{\pi k_c}, Z\right) = \Theta(Y, Z),$$

where πk_c is the critical wave number. In the next section we discuss the linear stability analysis of the system which is of great utility in the local nonlinear stability analysis to be discussed further on.

1.2 Linear Stability Analysis

It can easily be proved that the principle of exchange of stabilities (PES)is valid in the problem and hence we consider only the marginal stationary state. In order to make a linear stability analysis we consider the linear and steady-state version of Eqs. (31)-(32) and assume the solutions to be periodicwaves of the form:

$$\Psi(Y, Z) = \Psi_0 \sin(\pi k_c Y) \sin(\pi Z), \qquad (34)$$

$$\Theta(Y, Z) = \Theta_0 \cos(\pi k_c Y) \sin(\pi Z). \qquad (35)$$

The quantities Ψ_0 and Θ_0 are, respectively, amplitudes of the stream function and temperature. The normal mode solutions of Eqs. (34) and (35) satisfy the boundary / periodicity conditions in Eq. (33).

Following standard procedure, we can obtain the expression for the critical Rayleigh number and wave number in the form:

$$Ra_c = \frac{\delta_c^6}{\pi^2 k_c^2}\Lambda + \frac{\sigma^2}{\delta_c^2}, \quad k_c^2 = \frac{-(\Lambda\pi^2 + \sigma^2) + \sqrt{9\pi^4\Lambda^2 + 10\Lambda\pi^2\sigma^2 + \sigma^4}}{4\Lambda\pi^2}.$$

If $\Lambda = 1$ and $\sigma^2 = 0$, we get

$$\mathrm{Ra}_c = \frac{\delta_c^6}{\pi^2 k_c^2}, \quad k_c = \frac{1}{\sqrt{2}}, \tag{36}$$

where the critical Rayleigh number, Ra_c, indicates transition from linear to nonlinear instability and $\delta^2 = \pi^2(k_c^2 + 1)$. The linear theory predicts only the condition for the onset of convection and is silent about the heat transport. We now embark on a weakly nonlinear analysis by means of a truncated representation of Fourier series for stream function and temperature fields to find the effect of various parameters on finite-amplitude convection and to know the amount of heat transfer.

2 Weakly Nonlinear Stability Analysis

The first effect of nonlinearity is to distort the temperature field through the interaction of Ψ and Θ. The distortion of the temperature field will correspond to a change in the horizontal mean, i.e., a component of the form $\sin(2\pi z)$ will be generated. Substituting a minimal double Fourier series which describes the unsteady finite-amplitude convection in a Newtonian liquid given by

$$\Psi(Y, Z, \tau) = A(\tau)\sin(\pi k_c Y)\sin(\pi Z), \tag{37}$$

$$\Theta(Y, Z, \tau) = B(\tau)\cos(\pi k_c Y)\sin(\pi Z) - C(\tau)\sin(2\pi Z), \tag{38}$$

into Eqs. (31)-(32) and adopting the standard orthogonalization procedure for the Galerkin expansion, we obtain the following system of equations.

$$\frac{dA}{d\tau} = \Pr\left[\frac{-\pi k_c \mathrm{Ra}}{\delta_c^2} B(\tau) - (\Lambda\delta_c^2 + \sigma^2) A(\tau)\right],$$

$$\frac{dB}{d\tau} = -\pi k_c A(\tau) - \delta_c^2 B(\tau) + \pi^2 k_c A(\tau) C(\tau), \tag{39}$$

$$\frac{dC}{d\tau} = \frac{\pi^2 k_c}{2} A(\tau) B(\tau) - 4\pi^2 C(\tau).$$

If $\Lambda = 1$ and $\sigma^2 = 0$ in Eq. (39), we get the classical form of the Lorenzmodel:

$$\frac{dA}{d\tau} = \Pr\left[\frac{-\pi k_c \mathrm{Ra}}{\delta_c^2} B(\tau) - \delta_c^2 A(\tau)\right],$$

$$\frac{dB}{d\tau} = -\pi k_c A(\tau) - \delta_c^2 B(\tau) + \pi^2 k_c A(\tau)C(\tau), \qquad (40)$$

$$\frac{dC}{d\tau} = \frac{\pi^2 k_c}{2} A(\tau)B(\tau) - 4\pi^2 C(\tau).$$

3 Scaling

Let

$$A = k_1 A_1,\; B = k_2 B_1,\; C = k_3 C_1,\; \tau = \delta_c^2 \tau_1. \qquad (41)$$

Using Eq. (41) in Eq. (39), we get

$$\frac{dA_1}{d\tau_1} = \frac{\Pr}{\delta_c^2}\sigma^2\left(\Lambda + \frac{4}{\delta_c^2}\Lambda\sigma^2\right)\left(\frac{\pi k_c Ra}{\delta_c^2}\frac{k_2}{k_1}B_1 - A_1\right), \qquad (42)$$

$$\frac{dB_1}{d\tau_1} = -\frac{\pi k_c}{\delta_c^2}\frac{k_1}{k_2}A_1 - B_1 + \frac{\pi^2 k_c}{\delta_c^2}\frac{k_1 k_3}{k_2}A_1 C_1,$$

$$\frac{dC_1}{d\tau_1} = \frac{\pi^2 k_c}{2\delta_c^2}\frac{k_1 k_2}{k_3}A_1 B_1 - \frac{4\pi^2}{\delta_c^2}C_1.$$

We plan to bring Eq. (39) into the classical form of the Lorenz model:

$$\frac{dA}{d\tau} = \Pr\left(\Lambda + \frac{\sigma^2}{\delta_c^2}\right)(B - A),$$

$$\frac{dB}{d\tau} = r^* A - B - AC,$$

$$\frac{dC}{d\tau}$$

. (43)

$d\tau$ $= AB - \beta C.$

Comparing Eqs. (41) and (42), we now recognise that the following must hold:

$\pi k_c Ra$ k_2 δ_c δ δ $\pi^2 k_c$ $k_1 k_3$

4 $\Lambda +$ σ^2 $=$ 1, δ^2

k c 2

δ $C =$ 1,

2 k_2 (44)

c r c δ

r* $= \dfrac{r}{\quad}$, $= 1$, β $4\pi^2$,

$=$ 2

c c k ,

c

k

1

k

2

where σ^2 $2\delta^2$ k_3 c

Λ +

2

c

$\pi^2 k$ ^{2}Ra . (45)

$=$ 6

c δ_c 2

We now solve Eq. (44) for k_1, k_2 and k_3 to get

$$k_1 = \frac{\sqrt{2}\delta}{\pi^2 k_c}, \quad k_2 = \frac{\sqrt{2}\delta^5\,\sigma^2\left(\Lambda + \frac{\ }{\delta^2}\right)_c}{\pi^2 k_c^2 Ra}, \quad k_3 = -\frac{6\left(\Lambda + \frac{\sigma}{\delta^2}\right)_c}{\pi^3 k_c^2 Ra}. \tag{46}$$

Using Eq. (46) in Eq. (42), we get

$$\frac{dA_1}{d\tau_1} = \Pr^*(B_1 - A_1), \tag{47}$$

$$\frac{dB_1}{d\tau_1} = r^* A_1 - B_1 - A_1 C_1, \tag{48}$$

$$\frac{dC_1}{d\tau_1} = A_1 B_1 - \beta C_1, \tag{49}$$

where

$$\Pr^* = \frac{\Pr}{\sigma^2\left(\Lambda + \frac{\ }{\delta^2}\right)_c}, \quad r^* = \frac{r}{\frac{\ }{\sigma^2}\left(\Lambda + \frac{\ }{\delta^2}\right)_c}. \tag{50}$$

Equations (47)-(49) form a nonlinear autonomous system(generalized tri- modal Lorenz model) and A_1, B_1 are the amplitudes in normal mode solu- tion and C_1 is the amplitude of convective mode. It is well known in the problems as these that the trajectories of the solution of the Lorenz modelin phase-space remain within a bounded region. In the next section we show that this trapping region is, in fact, a sphere for the current problem.

If $\Lambda = 1$ and $\sigma^2 = 0$ in Eqs. (47)-(49), we get the classical form of the Lorenz model:

$$\frac{dA_1}{d\tau_1} = \Pr(B_1 - A_1),$$
$$\frac{dB_1}{d\tau_1} = rA_1 - B_1 - A_1C_1, \tag{51}$$
$$\frac{dC_1}{d\tau_1} = A_1B_1 - \beta C_1.$$

4 Trapping Region

Multiplying Eqs. (47) and (48) by A_1 and B_1 respectively, we get

$$A_1\frac{dA_1}{d\tau_1} = \Pr{}^*A_1(B_1 - A_1), \tag{52}$$

$$B_1\frac{dB_1}{d\tau_1} = r^*A_1B_1 - B_1^2 - A_1B_1C_1. \tag{53}$$

Adding Eqs. (52) and (53), we get

$$A_1 \frac{dA_1}{d\tau_1} + B_1 \frac{dB_1}{d\tau_1} = -\Pr^* A_1^2 - B_1^2 + A_1 B_1 [\Pr^* + r^* - C_1]. \quad (54)$$

To get an equation of a sphere from Eqs. (49) and (54), we multiply Eq. (49)by $[C_1 - \Pr^* - r^*]$ and add the resulting equation to Eq. (54). This gives us

$$\frac{dE}{d\tau_1} = A_1 \frac{dA_1}{d\tau_1} + B_1 \frac{dB_1}{d\tau_1} + [C_1 - \Pr^* - r^*] \frac{d}{d\tau_1}[C_1 - \Pr^* - r^*]. \quad (55)$$

Integrating the above equation , we get the trapping region in the form

$$E = \frac{1}{2} \left[A_1^2 + B_1^2 + (C_1 - \Pr^* - r^*)^2 \right]. \quad (56)$$

The post-onset trajectories of the Lorenz system (47)-(49) enter and stay within a sphere with center $(0, 0, \Pr^* + r^*)$ and radius $\sqrt{2}$ given by

$$A_1^2 + B_1^2 + (C_1 - \Pr^* - r^*)^2 = (\sqrt{2})^2. \quad (57)$$

Noting that the Lorenz model is, in general, not analytically tractable we now move on to derive the analytically tractable Ginzburg-Landau equation from the tri-modal Lorenz model.

If $\Lambda = 1$ and $\sigma^2 = 0$ in Eq. (57), we get the spherical classical form of the Lorenz model:

$$A_1^2 + B_1^2 + (C_1 - \Pr - r)^2 = (\sqrt{2})^2. \quad (58)$$

5 Ginzburg-Landau Amplitude Equation fromthe Lorenz model

From the Eqs. (45) and (46) B_1 and C_1 can be obtained in terms of A_1 as:

$$B_1 = \frac{1}{1} \frac{dA_1}{d\tau_1} + A_1, \quad (59)$$

$$C_1 = -\frac{1}{A_1} \left[\frac{1}{\Pr^*} \frac{d^2 A_1}{d\tau_1^2} + A_1 \right]. \quad (60)$$

$\Pr^*$

$d\tau^2$ $\quad d\tau_1 \quad 1 \;\; dA_1$

$\underline{d}$ $\quad + \text{——} \; \text{——} \; \text{—} \, r \;\; A_1 + \Pr^* \, d\tau_1$

$\underline{A_1} \quad *$

$d\tau_1$

Substituting Eqs. (54) and (55) in Eq. (47), we get a third order differential equation in A_1. Neglecting the terms of the type $\frac{d^3A_1}{d\tau_1{}^3}$, $\left(\frac{dA_1}{d\tau_1}\right)^2$,

$\frac{d^2A_1}{}$ $\quad \underline{dA_1} \quad \frac{d^2A_1}{}$

$A_1 \;\; d\tau_1$ an $d\tau_1 \;\; d\tau_1$, we get the Ginzburg-Landau model in the form 2 d 2

$$\frac{dA_1}{d\tau_1} = \frac{\Pr^* \;\; 1}{1 + \Pr^*} [\beta(r^* - 1)A_1 - A_1^3]. \quad \beta \qquad (61)$$

Equation (56) is a Bernoulli equation in A_1 which can be solved using an initial condition $A(0)=A_0$ and the solution is given by

$$A_1 = s\,\frac{Q_3}{Q + (Q\;A^{-2} - Q\;)e^{-2Q_3\tau_1}}, \qquad (62)$$

$2 \qquad = \qquad 0 \qquad 2$

where $\quad 2 \quad \overline{1 + \Pr^*}$

$= \beta(r^* - 1), \;\; Q \quad 3 \quad \Pr^*$

$Q \quad 1$

$\beta \quad 3 \quad \frac{1}{}, Q$

$$= (r^* - 1)\frac{\Pr^*}{1+\Pr^*}.$$

It is one of the intentions of the project to study the pre-onset and post-onset critical points of the tri-modal Lorenz model and these are considered in the succeeding section.

If $\Lambda = 1$ and $\sigma^2 = 0$ in Eq. (62), we get the classical form

$$A_1 = \sqrt{\frac{Q_3}{Q_2 + (Q_3 A_0^{-2} - Q_2)e^{-2Q_3\tau_1}}}, \tag{63}$$

where

$$Q_1 = \beta(r-1),\ Q_2 = \frac{\Pr}{1+\Pr}\frac{1}{\beta},\ Q_3 = (r-1)\frac{\Pr}{1+\Pr}.$$

6 Steady Finite Amplitude Convection

We note that the nonlinear system of autonomous differential equations (47)- (49) is not amenable to analytical treatment for the general time-dependent variables and it is to be solved by means of a numerical method. However, in the case of steady motions, these equations can be solved in closed form.

The solution of the system (47)-(49) with left hand sides omitted is

$$(0,0,0),\ (\pm\sqrt{\beta(r^*-1)}, \pm\sqrt{\beta(r^*-1)}, (r^*-1)). \tag{64}$$

These are the post-onset critical points of the dynamical system (47)-(49). The solution $A_1 = B_1 = C_1 = 0$ of the Lorenz model represents the state of no convection and non-zero values represent the convective state. Following standard procedure with the linear system of autonomous differential equa- tions, it can be easily shown that the only pre-onset critical point is (0, 0, 0) which is a saddle point. In the next section we quantify the Hopf-BifurcationRayleigh number.

If $\Lambda = 1$ and $\sigma^2 = 0$ in Eq. (64), we get the classical form

$$(0,0,0),\ (\pm\sqrt{\beta(r-1)}, \pm\sqrt{\beta(r-1)}, (r-1)). \tag{65}$$

7 Hopf-Bifurcation Rayleigh Number

Linearization of the Lorenz equations (47)-(49) about $(\bar{X}, \bar{Y}, \bar{Z})$ yields:

$$\begin{pmatrix}\dot{X}\\ \dot{Y}\\ \dot{Z}\end{pmatrix} = \begin{pmatrix} -Pr^* & Pr^* & 0\\ r^*-\bar{Z} & -1 & -\bar{X}\\ \bar{Y} & \bar{X} & -\beta \end{pmatrix}\begin{pmatrix}X\\ Y\\ Z\end{pmatrix} \tag{66}$$

To get the eigenvalues of the (3×3) coefficient matrix, we consider

$$\begin{vmatrix} -Pr^*-\lambda & Pr^* & 0\\ r^*-\bar{Z} & -1-\lambda & -\bar{X}\\ \bar{Y} & \bar{X} & -\beta-\lambda \end{vmatrix} = 0.$$

Expanding the determinant gives us

$$\lambda^3 + (1+\beta+Pr^*)\lambda^2 + (\beta + Pr^* + Pr^*\beta + \bar{X}^2 - Pr^*r^* + Pr^*\bar{Z})\lambda$$
$$+(Pr^*\beta + Pr^*\bar{X}^2 - Pr^*r^*\beta + Pr^*\bar{Z}\beta + Pr^*\bar{X}\bar{Y}) = 0. \tag{67}$$

If we take $(\bar{X}, \bar{Y}, \bar{Z})$ to be the equilibrium point $(0, 0, 0)$, we get

$$\lambda^3 + (1+\beta+Pr^*)\lambda^2 + (\beta + Pr^* + Pr^*\beta - Pr^*r^*)\lambda + (Pr^*\beta - Pr^*r^*\beta) = 0.$$
(68)

A root of Eq. (68) is β and so we can factorize Eq. (68) to get

$(\lambda+\beta)\,[\lambda^2 + (1+Pr^*)\lambda + Pr^*(1-r^*)] = 0.$

The roots(eigenvalues) of Eq. (68) are

$$\lambda_1 = \frac{-(1+Pr^*) + \sqrt{(1+Pr^*)^2 - 4Pr^*(1-r^*)}}{2},$$

$$\lambda_2 = \frac{-(1+Pr^*) - \sqrt{(1+Pr^*)^2 - 4Pr^*(1-r^*)}}{2}$$

,

(1 + Pr*) (1 + 2

Pr*)^2 4Pr*(1 r*)

$\lambda_3 = -\beta.$

Putting $\lambda = i\mu$ in Eq. (68), we get

$-i\mu^3 - (\beta + \Pr^* + 1)\mu^2 + (\beta + \beta\Pr^* + \Pr^* - \Pr^* r^*)i\mu + \beta\Pr^*(1 - r^*) = 0.$ (69) The real and imaginary parts of Eq. (69) are:

$$\mu^3 = -(\beta + \Pr^* + \Pr^*\beta - \Pr^* r^*)\mu. \qquad (70)$$

$-(1 + \Pr^* + \beta)\mu^2 + \Pr^*\beta(1 - r^*) = 0,$) .

Eliminating μ^2 between the two equations in Eq. (70), we get an expression for the Hopf-bifurcation Rayleigh number in the form:

$\Pr^*(\Pr^* + \beta + 3)(\Lambda + {}^{\sigma}2)$

$r_H =$ $\frac{c}{\delta}$. (71)

2

$(\Pr^* - \beta - 1)$

8 Results and Discussion

Regular and chaotic convective motions are considered in the problem of Rayleigh-Bénard -Brinkman convection. The linear stability analysis of the system yields information on the onset of regular motion. A weakly nonlinearstability analysis provides information on the onset of chaotic motion. The critical wave number, k_c, and Rayleigh number, Ra_c, of regular convective motion are given by Eq. (36). The Hopf-bifurcation Rayleigh number of chaotic motion, r_H, is given by Eq. (71). Table 1 documents the values of k_c, Ra_c, Pr^* and r_H for three different values of σ^2. It is clear from the table that onset of regular convective motions in the presence of porus medium is delayed when compared with that in its absence. A similar observatipn is true of the onset of chaotic motions.

9 References

S Chandrasekhar. Hydrodynamic and hydromagnetic stability. Claren- don Press, London. 1961.

J K Platten and J C Legrose. Convection in Liquids. Springer-Verlag,Berlin-Heidelberg-New York-Tokyo. 1984.

B Straughan. The energy method, stability and nonlinear convection.Springer, London. 2003.

P G Drazin and W H Reid. Hydrodynamic stability. Cambridge uni-versity press, Cambridge. 2004.

D A Nield and A Bejan. Convection in porous media, 3rd Ed., volume

3. Springer, USA, 2006.

R K Zeytounian. Convection in Fluids. Fluid Mechanics and its Ap-plications 90. Springer, Netherlands, 2009.

K Vafai. Handbook of porous media. CRC Press, Boca Raton, 2015.